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Non-commutative L^p-spaces

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0. Introduction

Non-commutative L^p-spaces are, by definition, a family of Banach spaces $L^p(\mathcal{M})$ which "interpolates" a von Neumann algebra $\mathcal{M}$ and its unique predual $\mathcal{M}_*$ in a certain sense. If a von Neumann algebra $\mathcal{M}$ is commutative, then there exists a measure space (X, μ) such that the pair $(\mathcal{M}, \mathcal{M}_*)$ is identified with $(L^\infty(X, \mu), L^1(X, \mu))$. Hence in general cases, von Neumann algebras are called "non-commutative L^p-spaces" and their preduals are "non-commutative L^1-spaces", and in any construction, non-commutative L^p-spaces constructed for a commutative von Neumann algebra should reduce to usual L^p-spaces.

The construction of non-commutative L^p-spaces has been made by Haagerup ([**Ha**]), Hilsum ([**Hi**]), Araki-Masuda ([**AM**]), Kosaki ([**Ko**]) and Terp ([**Te**]). All of these non-commutative L^p-spaces are equivalent in the sense of being isometrically isomorphic, but the constructions are quite different.

In [**Ko**] and [**Te**], the tool for constructing non-commutative L^p-spaces is Calderón's complex interpolation method ([**Ca**], [**BL**]). The complex interpolation method is a way in harmonic analysis to produce a one-parameter family of Banach spaces $C_\theta(A_0, A_1)$, $0 < \theta < 1$ from a compatible pair (A_0, A_1) of Banach spaces.

Consider the abelian case. It is known that if we apply the complex interpolation method to the compatible pair $(L^\infty(X, \mu), L^1(X, \varphi))$, where (X, μ) is a measure space, we obtain the family $\{L^p(X, \mu)\}_{1<p<\infty}$ as the interpolation spaces. Here, a pair (A_0, A_1) of Banach spaces is called compatible if both A_0 and A_1 are embedded into some linear space E with a common part which is dense in both A_0 and A_1. In the abelian case, $L^\infty(X, \mu)$, and $L^1(X, \mu)$ are naturally embedded into the Banach space $L^\infty(X, \mu) + L^1(X, \mu)$ endowed with the infimum norm, in which a weak* dense subspace of $L^\infty(X, \mu)$ and a norm dense subspace of $L^1(X, \mu)$ is identified. But in the general case, it is difficult to construct embeddings of the pair $(\mathcal{M}, \mathcal{M}_*)$ such that it is compatible.

Kosaki ([**Ko**]) considered the complex interpolation of the pair $(\mathcal{M}, \mathcal{M}_*)$ with a fixed faithful normal state φ on $\mathcal{M}$. and he constructed a compatible pair $(\mathcal{M}, \mathcal{M}_*)$ by the embedding

$$\begin{array}{ll} & x \in \mathcal{M} \to x\varphi \in \mathcal{M}_*, \\ (\textit{resp.} & x \in \mathcal{M} \to \varphi x \in \mathcal{M}_*. \quad) \end{array}$$

The non-commutative L^p-spaces (called "left L^p-spaces" *resp.* "right L^p-spaces") are defined as the interpolation spaces of the compatible pair $(\mathcal{M}, \mathcal{M}_*)$. Kosaki discussed more generally a one-parameter family $(\mathcal{M}, \mathcal{M}_*)_\eta$, $-1/2 \leq \eta \leq 1/2$ of compatible pairs, in each case $\mathcal{M}$ is embedded into $\mathcal{M}_*$, and showed the equivalence to Haagerup's L^p-spaces.

Kosaki's construction is sufficient for dealing with von Neumann algebras with faithful normal states, but in general, weights appears naturally such as the dual weights for the crossed products, and the canonical weights on the group algebras. Making $(\mathcal{M}, \mathcal{M}_*)$ into a compatible pair for a fixed faithful normal semi-finite weight φ on $\mathcal{M}$ is much more difficult than the state case because of the absence of such embeddings. Terp ([**Te**]) constructed a compatible pair $(\mathcal{M}, \mathcal{M}_*)$ for a faithful normal semi-finite weight φ, which is equal to the "central" case in Kosaki's construction, and showed the equivalence to Hilsum's L^p-spaces If we can extend Terp's construction to a one-parameter family as in Kosaki's, then we can use unified arguments for the non-commutative L^p-spaces.

In this paper, for a faithful normal semi-finite weight φ on $\mathcal{M}$, we construct a complex one-parameter family of compatible pairs $(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}$, $\alpha \in \mathbb{C}$. and obtain non-commutative L^p-spaces $L^p_{(\alpha)}(\varphi)$ by the complex interpolation method. This construction is a generalization of Kosaki's one and Terp's one. When α is real, $|\alpha| \leq 1/2$ and φ is a state, then the compatible pair $(\mathcal{M}, \mathcal{M}_*)_\alpha$ is the same as Kosaki's. When $\alpha = 0$, the compatible pair is the same as Terp's. Next, we construct isometric isomorphisms

$$U_{p,(\alpha,\beta)} : L^p_{(\alpha)}(\varphi) \to L^p_{(\beta)}(\varphi)$$

for $\alpha, \beta \in \mathbb{C}$ and $1 < p < \infty$. Hence the families $\{L^p_{(\alpha)}\}_{1<p<\infty}, \alpha \in \mathbb{C}$, are equivalent to each other, in particular, to Terp's L^p-spaces.

Now, we summarize the content of each section.

In Section 1, we will explain Calderón's complex interpolation method. We define not only usual complex interpolation but another complex interpolation method, which is a modification of the original method inspired by [**Ko**] to deal with the σ-weak topology on von Neumann algebras.

In Section 2, we will devote to construction of non-commutative L^p-spaces by the complex interpolation method. Analytic elements for the modular operator Δ_φ play an important rôle in our discussion. Next, we determine the common part of $\mathcal{M}$ and $\mathcal{M}_*$.

Finally in Section 3, we show that each family $\{L^p_{(\alpha)}(\varphi)\}$ are equivalent, that is, the L^p-spaces for any parameter $\alpha \in \mathbb{C}$ are all isometrically isomorphic.

1. The Complex Interpolation Method

First of all, we will explain the usual complex interpolation

The pair of Banach spaces (A_0, A_1) is called *compatible* if there exists a normed space $\mathcal{E}$ such that both A_0 and A_1 can be embedded continuously into $\mathcal{E}$.

Let $A = (A_0, A_1)$ be compatible Banach spaces. We define a subspace $\Sigma(A)$ of $\mathcal{E}$ by $\Sigma(A) = A_0 + A_1$, and endow its norm by

$$\|a\|_{\Sigma(A)} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} | a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\},\ a \in \Sigma(A).$$

Then we define

$$\mathcal{F}(A) = \left\{ f : D \to \Sigma(A) \;\middle|\; \begin{array}{l} f \text{ is continuous, bounded on } D \\ \text{and holomorphic in the interior of } D \\ \text{and satisfies} \\ (1)\ f(it) \in A_0 \text{ for all } t \in \mathbb{R}, \\ \text{the function } t \in \mathbb{R} \mapsto f(it) \in A_0 \text{ is continuous} \\ \text{and } \lim_{t\to\pm\infty} \|f(it)\|_{A_0} = 0 \\ (2)\ f(1+it) \in A_1 \text{ for all } t \in \mathbb{R}, \\ \text{the function } t \in \mathbb{R} \mapsto f(1+it) \in A_1 \text{ is continuous} \\ \text{and } \lim_{t\to\pm\infty} \|f(1+it)\|_{A_1} = 0, \end{array} \right\}$$

where D means the closed strip $\{\alpha \in \mathbb{C} | 0 \leq \operatorname{Re} \alpha \leq 1\}$ We endow the norm of $\mathcal{F}(A)$ by

$$\|f\|_{\mathcal{F}(A)} = \max\{\max_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \max_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\},\ f \in \mathcal{F}(A).$$

This is indeed a norm of $\mathcal{F}(A)$ by the Phragmén-Lindelöf theorem. Then we define the interpolation spaces by

$$C_\theta(A) = \{a \in \Sigma(A) | a = f(\theta) \text{ for some } f \in \mathcal{F}(A)\}$$

with its norm

$$\|a\|_{C_\theta(A)} = \inf\{\|f\|_{\mathcal{F}(A)} | f(\theta) = a\},\ a \in C_\theta(A)$$

for each θ, $0 < \theta < 1$. It is easy to see that $\mathcal{F}(A)$ and A_θ, $0 < \theta < 1$, are Banach spaces.

Now, we introduce another complex interpolation method, which will be used in Section 3 to prove equivalence of non-commutative L^p-spaces.

For a compatible pair $A = (A_0, A_1)$, we define $\Sigma(A)$ as above, and fix a $\sigma(\Sigma(A)^*, \Sigma(A))$-dense subspace Π of $\Sigma(A)^*$. Then we define $\mathcal{F}'(A)$ by

$$\mathcal{F}'(A) = \left\{ f : D \to \Sigma(A) \;\middle|\; \begin{array}{l} f \text{ is } \sigma(\Sigma(A), \Pi)\text{-continuous, bounded on } D \\ \text{and holomorphic in the interior of } D \\ \text{and satisfies} \\ (1)\ f(it) \in A_0 \text{ for all } t \in \mathbb{R}, \\ \text{and } \sup_{t\in\mathbb{R}} \|f(it)\|_{A_0} < \infty \\ (2)\ f(1+it) \in A_1 \text{ for all } t \in \mathbb{R}, \\ \text{the function } t \in \mathbb{R} \mapsto f(1+it) \in A_1 \text{ is norm continuous} \\ \text{and } \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1} < \infty. \end{array} \right\}$$

Then we set

$$\|f\|_{\mathcal{F}'(A)} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in R} \|f(1+it)\|_{A_1}\},\ f \in \mathcal{F}'(A).$$

Clearly, $\mathcal{F}(A)$ is a closed subspace of $\mathcal{F}'(A)$. We define interpolation spaces $C'_\theta(A)$, $0 < \theta < 1$ via $\mathcal{F}'(A)$ in a similar fashion.

2. Construction of Non-commutative L^p-space

Let $\mathcal{M}$ be a von Neumann algebra and φ be a faithful normal semi-finite weight on $\mathcal{M}$. We will construct a complex one-parameter family of the families $\{L^p_{(\alpha)}(\varphi)\}_{1\le p\le\infty}$, $\alpha \in \mathbb{C}$, of non-commutative L^p-spaces by the complex interpolation method.

Let $\{\pi_\varphi, n_\varphi, \Lambda\}$ be the GNS construction induced from $(\mathcal{M}, \varphi)$, $\mathfrak{A}, \mathfrak{A}_0$ be the associated left Hilbert algebra and the Tomita algebra (see **[Ta]**), respectively. By identifying $\mathcal{M}$ and $\pi_\varphi(\mathcal{M})$, we write x instead of $\pi_\varphi(x)$.

Now, for each $\alpha \in \mathbb{C}$, we define

$$L_{(\alpha)} = \left\{ x \in \mathcal{M} \;\middle|\; \begin{array}{l} \text{there exists a functional } \varphi_x^{(\alpha)} \in \mathcal{M}_* \text{ such that} \\ \varphi_x^{(\alpha)}(y^* z) = (xJ\Delta^{\overline{\alpha}}\Lambda(y) | J\Delta^{-\alpha}\Lambda(z)) \\ \text{for all } y, z \in \mathfrak{a}_0, \end{array} \right\}$$

where $\mathfrak{a}_0$ means $\Lambda^{-1}(\mathfrak{A}_0)$ and J and Δ are the modular conjugation and the modular operator, respectively.

The following proposition is easily proved by the density of the Tomita algebra.

Proposition 2.1. For each $\alpha \in \mathbb{C}$, $L_{(\alpha)}$ is a linear manifold with

$$\varphi^{(\alpha)}_{\lambda x + \mu y} = \lambda \varphi_x^{(\alpha)} + \mu \varphi_y^{(\alpha)},\ \lambda, \mu \in \mathbb{C},\ x, y \in L_{(\alpha)}.$$

We define the norm of $L_{(\alpha)}$ by

$$\|x\|_{L_{(\alpha)}} = \max\{\|x\|_\infty, \|\varphi_x^{(\alpha)}\|_1\},$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the norms of $\mathcal{M}$ and $\mathcal{M}_*$, respectively. We note that $L_{(\alpha)}$ is a Banach space.

The next proposition shows that $L_{(\alpha)}$ contains enough elements.

Proposition 2.2. Let $\mathfrak{a}_0^2$ be the algebraic linear span of the elements of the form y^*z, $y, z \in \mathfrak{a}_0$. Then, for any $\alpha \in \mathbb{C}$, we have

$$\mathfrak{a}_0^2 \subset L_{(\alpha)}$$

and

$$\varphi_{y^*z}^{(\alpha)} = \omega_{J\Delta^{-\overline{\alpha}}\Lambda(y),\ J\Delta^\alpha\Lambda(z)},$$

where $\omega_{\xi,\eta}$, $\xi, \eta \in \mathcal{H}_\varphi$ means the vectorial functional $(\cdot\ \xi|\eta)$.

Proof)

Bearing the fact

$$\pi_r(J\Delta^{\overline{\alpha}}\Lambda(z))\xi = J\Delta^{\overline{\alpha}}zJ\Delta^\alpha\xi,\ \xi \in \mathfrak{A},\ \alpha \in \mathbb{C}$$

in mind, we compute, for $x, y, z, w \in \mathfrak{a}_0$,

$$\begin{aligned}
(y^*zJ\Delta^{\overline{\alpha}}\Lambda(x)|J\Delta^{-\alpha}\Lambda(w)) &= (zJ\Delta^{\overline{\alpha}}\Lambda(x)|yJ\Delta^{-\alpha}\Lambda(w)) \\
&= (\pi_r(J\Delta^{\overline{\alpha}}\Lambda(x))\Lambda(z)|\pi_r(J\Delta^{-\alpha}\Lambda(w))\Lambda(y)) \\
&= (J\Delta^{\overline{\alpha}}xJ\Delta^\alpha\Lambda(z)|J\Delta^{\overline{\alpha}}wJ\Delta^\alpha\Lambda(y)) \\
&= (\Delta^{\overline{\alpha}}wJ\Delta^\alpha\Lambda(y)|\Delta^{\overline{\alpha}}xJ\Delta^\alpha\Lambda(z)) \\
&= (wJ\Delta^\alpha\Lambda(y)|xJ\Delta^\alpha\Lambda(z)) \\
&= (x^*wJ\Delta^\alpha\Lambda(y)|J\Delta^\alpha\Lambda(z)).
\end{aligned}$$

Hence we get $y^*z \in L_{(\alpha)}$ and that

$$\varphi_{y^*z}^{(\alpha)} = \omega_{J\Delta^{-\overline{\alpha}}\Lambda(y),J\Delta^\alpha\Lambda(z)}. \quad \square$$

Now we define two maps: $i_{(\alpha)} : L_{(\alpha)} \to \mathcal{M}$ is a canonical inclusion and $j_{(\alpha)} : L_{(\alpha)} \to \mathcal{M}_*$ is defined by $j_\alpha(x) = \varphi_x^{(\alpha)}, x \in L_{(\alpha)}$. From the density of the Tomita algebra, the above two maps are both norm-decreasing and injective.

Furthermore we have

Proposition 2.3.

(a) The set $i_{(\alpha)}(L_{(\alpha)})$ is σ-weakly dense in $\mathcal{M}$.

(b) The set $j_{(\alpha)}(L_{(\alpha)})$ is norm dense in $\mathcal{M}_*$.

Proof)

(a) It is easily proved by Prop 2.2.

(b) Suppose that the norm closure of $j_{(\alpha)}(L_{(\alpha)})$ is not equal to $\mathcal{M}_*$. Since $(\mathcal{M}_*)^*$=M, there exists non-zero $x_0 \in \mathcal{M}$ which vanishes on $j_\alpha(L_{(\alpha)})$ by the Hahn-Banach theorem. Rephrasing this, we have

$$(x_0J\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{-\alpha}\Lambda(z)) = 0$$

for all $y, z \in \mathfrak{a}_0$. On the other hand, $J\Delta^\beta\mathfrak{a}_0 = \mathfrak{a}_0$, $\beta \in \mathbb{C}$, is dense in $\mathcal{H}_\varphi$, so that $x_0 = 0$, hence a contradiction. $\square$

Next, we define adjoint maps $i^*_{(-\alpha)} : \mathcal{M}_* \to L^*_{(-\alpha)}$ and $j^*_{(-\alpha)} : \mathcal{M} \to L^*_{(-\alpha)}$ of $i_{(-\alpha)}$ and $j_{(-\alpha)}$: $i^*_{(-\alpha)}$ is the restriction to $\mathcal{M}_*$ of the canonical adjoint map of $i_{(-\alpha)}$, and $j^*_{(-\alpha)}$ is the canonical adjoint map. Explicitly,

$$\begin{aligned} \langle y, i^*_{(\alpha)}(\psi)\rangle_{L_{(-\alpha)},L^*_{(-\alpha)}} &= \psi(y) && ,y \in L_{(-\alpha)},\ \psi \in \mathcal{M}_*; \\ \langle y, j^*_{(\alpha)}(x)\rangle_{L_{(-\alpha)},L^*_{(-\alpha)}} &= \varphi^{(-\alpha)}_y(x) && ,y \in L_{(-\alpha)},\ x \in \mathcal{M}. \end{aligned}$$

The maps $i^*_{(-\alpha)}$ and $j^*_{(-\alpha)}$ are also norm-decreasing and Proposition 2.3 tells us that they are injective. We call $(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}$ the compatible pair obtained from the above considerations, and set non-commutative L^p-spaces by

$$L^p_{(\alpha)}(\mathcal{M}, \varphi) = C_{1/p}(\mathcal{M}, \mathcal{M}_*)_{(\alpha)},\ 1 < p < \infty,\ \alpha \in \mathbb{C}.$$

The notation $L^p_{(\alpha)}(\mathcal{M}, \varphi)$ will be often abbreviated as $L^p_{(\alpha)}(\varphi)$ in this paper.

Now, our next aim is to prove the following theorem:

Theorem 2.4. The diagram

$$\begin{tikzcd} & \mathcal{M} \arrow[dr, "i^*_{(-\alpha)}"] & \\ L_{(\alpha)} \arrow[ur, "i_{(\alpha)}"] \arrow[dr, "j_{(\alpha)}"'] & & L^*_{(-\alpha)} \\ & \mathcal{M}_* \arrow[ur, "j^*_{(-\alpha)}"'] & \end{tikzcd}$$

is commutative, that is, the formula

$$\varphi^{(\alpha)}_x(y) = \varphi^{(-\alpha)}_y(x),\ x \in L_{(\alpha)},\ y \in L_{(-\alpha)}.$$

holds.

We will divide the proof of Theorem 2.4 into several propositions.

Proposition 2.5.

$$\varphi^{(\alpha)}_{y^*z}(x^*w) = \varphi^{(-\alpha)}_{x^*w}(y^*z)$$

for all $x, y, z, w \in \mathfrak{a}_0$.

Proof)

Using Proposition 2.2 together with the definition of $L_{(\alpha)}$, we have

$$\begin{aligned} \varphi^{(\alpha)}_{y^*z}(x^*w) &= \omega_{J\Delta^{-\overline{\alpha}}\Lambda(y),\ J\Delta^{\alpha}\Lambda(z)}(x^*w) \\ &= (x^*wJ\Delta^{-\overline{\alpha}}\Lambda(y)|J\Delta^{\alpha}\Lambda(z)) \\ &= \varphi^{(-\alpha)}_{x^*w}(y^*z).\ \square \end{aligned}$$

The following proposition shows that $L_{(\alpha)}$ is a $\mathfrak{a}_0$-bimodule.

Proposition 2.6. (a) For each $\alpha \in \mathbb{C}$, the Banach space $L_{(\alpha)}$ is $(\mathfrak{a}_0, \mathfrak{a}_0)$-invariant, that is, for $a, b \in \mathfrak{a}_0$ and $x \in L_{(\alpha)}$, $axb \in L_{(\alpha)}$ and

$$\varphi^{(\alpha)}_{axb} = \sigma^{\varphi}_{-i\alpha-i/2}(a)\varphi^{(\alpha)}_x \sigma^{\varphi}_{-i\alpha+i/2}(b),$$

where the symbol $u\psi v$, $u, v \in \mathcal{M}$, $\psi \in \mathcal{M}_*$, means $\langle u\psi v, a\rangle_{\mathcal{M}_*,\mathcal{M}} = \psi(vau)$, $a \in \mathcal{M}$.

(b) If $\operatorname{Re}\alpha = \operatorname{Re}\beta$, then $L_{(\alpha)} = L_{(\beta)}$ and

$$\varphi^{(\alpha)}_x = \varphi^{(\beta)}_{\sigma^{\varphi}_{-t}(x)} = \varphi^{(\beta)}_x \circ \sigma^{\varphi}_t,$$

where $t = i(\beta - \alpha)$, for all $x \in L_{(\alpha)} = L_{(\beta)}$.

Proof)

(a) If we set

$$a' = \sigma_{-i\alpha-i/2}(a),\ b' = \sigma_{-i\alpha+i/2}(b),$$

then we have

$$\begin{aligned}\Lambda(a') &= \Delta^{\alpha+1/2}\Lambda(a)\\ &= J\Delta^{-\overline{\alpha}-1/2}J\Lambda(a)\\ &= J\Delta^{-\overline{\alpha}}J\Delta^{-1/2}\Lambda(a)\\ &= J\Delta^{-\overline{\alpha}}\Lambda(a^*)\end{aligned}$$

and that

$$\begin{aligned}b' &= \pi_l(\Delta^{\alpha-1/2}\Lambda(b))\\ &= \pi_l(J\Delta^{1/2}J\Delta^{\alpha}\Lambda(b))\\ &= \pi_l(J\Delta^{\alpha}\Lambda(b))^*.\end{aligned}$$

Thus we compute

$$\begin{aligned}&(axbJ\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{-\alpha}\Lambda(z))\\ &= (xbJ\Delta^{\overline{\alpha}}\Lambda(y)|a^*J\Delta^{-\alpha}\Lambda(z))\\ &= (x\Delta^{-\alpha}JJ\Delta^{\alpha}bJ\Delta^{\overline{\alpha}}\Lambda(y)|\Delta^{\overline{\alpha}}JJ\Delta^{-\overline{\alpha}}a^*J\Delta^{-\alpha}\Lambda(z))\\ &= (x\Delta^{-\alpha}J\pi_r(J\Delta^{\alpha}\Lambda(y))\Lambda(b)|J\Delta^{-\alpha}\pi_r(J\Delta^{-\overline{\alpha}}\Lambda(a^*))\Lambda(z))\\ &= (xJ\Delta^{\overline{\alpha}}yJ\Delta^{\alpha}\Lambda(b)|J\Delta^{-\alpha}zJ\Delta^{-\overline{\alpha}}\Lambda(a^*))\\ &= (xJ\Delta^{\overline{\alpha}}\Lambda(yb')|J\Delta^{-\alpha}\Lambda(za'))\\ &= \varphi^{(\alpha)}_x(b'y^*za')\\ &= \langle a'\varphi^{\alpha}_x b', y^*z\rangle_{\mathcal{M}_*,\mathcal{M}}.\end{aligned}$$

Hence, $axb \in L_{(\alpha)}$ and

$$\varphi^{(\alpha)}_{axb} = a'\varphi^{(\alpha)}_x b'.$$

(b) For $x \in L_{(\alpha)}$, we have

$$\begin{aligned}(xJ\Delta^{\overline{\beta}}\Lambda(y)|J\Delta^{-\beta}\Lambda(z)) &= (xJ\Delta^{it}\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{it}\Delta^{-\alpha}\Lambda(z))\\ &= (x\Delta^{it}J\Delta^{\overline{\alpha}}\Lambda(y)|\Delta^{it}J\Delta^{-\alpha}\Lambda(z))\\ &= (\Delta^{-it}x\Delta^{it}J\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{-\alpha}\Lambda(z))\\ &= (\sigma^{\varphi}_{-t}(x)J\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{-\alpha}\Lambda(z)).\end{aligned}$$

Hence $x \in L_{(\beta)}$ and

$$\varphi_x^{(\beta)} = \varphi^{(\alpha)}_{\sigma^\varphi_{-t}(x)}.$$

On the other hand,

$$\begin{aligned}(xJ\Delta^{\overline{\alpha}}\Delta^{it}\Lambda(y)|J\Delta^{-\alpha}\Delta^{it}\Lambda(z)) &= (xJ\Delta^{\overline{\alpha}}\Lambda(\sigma_t^\varphi(y))|J\Delta^{-\alpha}\Lambda(\sigma_t^\varphi(z)) \\ &= \varphi_x^{(\alpha)}(\sigma_t^\varphi(y^*z)).\end{aligned}$$

Hence $\varphi_x^{(\beta)} = \varphi_x^{(\alpha)} \circ \sigma_t^\varphi$. □

Proposition 2.7. Let $a, b \in \mathcal{M}$, and let $w_j, v_j (j = 1, 2, 3)$ be elements of $\mathfrak{a}_0$. Then, for all $t \in \mathbb{R}$, both $w_1^* w_2 a w_3$ and $v_1^* v_2 b v_3$ belong to $L_{(it)}$ and we have the equality

$$\varphi^{(it)}_{w_1^* w_2 a w_3}(v_1^* v_2 b v_3) = \varphi^{(-it)}_{v_1^* v_2 b v_3}(w_1^* w_2 a w_3)$$

Proof)

For $y, z \in \mathfrak{a}_0$, we have

$$\begin{aligned}(w_1^* w_2 a w_3 J\Lambda(y)|J\Lambda(z)) &= (w_2 a w_3 J\Lambda(y)|w_1 J\Lambda(z)) \\ &= (\pi_r(J\Lambda(y))\Lambda(w_2 a w_3)|\pi_r(J\Lambda(z))\Lambda(w_1)) \\ &= (JyJ\Lambda(w_2 a w_3)|JzJ\Lambda(w_1)) \\ &= (zJ\Lambda(w_1)|yJ\Lambda(w_2 a w_3)) \\ &= (y^* z J\Lambda(w_1)|J\Lambda(w_2 a w_3))\end{aligned}$$

Hence $w_1^* w_2 a w_3 \in L_{(0)}$ and

$$\varphi^{(0)}_{w_1^* w_2 a w_3} = \omega_{J\Lambda(w_1),\, J w_2 a \Lambda(w_3)}.$$

By Proposition 2.6(b) we have $w_1^* w_2 a w_3 \in L_{(it)}$ and

$$\begin{aligned}\varphi^{(it)}_{w_1^* w_2 a w_3} &= \omega_{J\Lambda(w_1),\ J w_2 \Lambda(w_3)} \circ \sigma_t^\varphi \\ &= \omega_{\Delta^{-it} J\Lambda(w_1),\ \Delta^{-it} J w_2 a \Lambda(w_3)}.\end{aligned}$$

Next we will prove the equality

$$\varphi^{(it)}_{w_1^* w_2 a w_3}(v_1^* v_2 b v_3) = \varphi^{(-\alpha)}_{v_1^* v_2 b v_3}(w_1^* w_2 a w_3).$$

For $r > 0$, let

$$(v_2 b v_3)_r = \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} \sigma_t^\varphi(v_2 b v_3) dt.$$

Then we have $(v_2 b v_3)_r \in \mathfrak{a}_0$,

$$(v_2 b v_3)_r \longrightarrow v_2 b v_3 \quad \sigma\text{-strongly}^*$$

and

$$\|\Lambda(v_2 b v_3) - \Lambda((v_2 b v_3)_r)\|_{\mathcal{H}_\varphi} \longrightarrow 0 \text{ as } r \longrightarrow +\infty.$$

Hence we compute

$$\begin{aligned}\varphi_(w_1^* w_2 a w_3)^{(it)}(v_1^*(v_2 b v_3)_r) &= (w_1^* w_2 a w_3 J\Delta^{-it}\Lambda(v_1)|J\Delta^{-it}\Lambda((v_2 b v_3)_r)) \\ &= \omega_{J\Lambda(v_1),\ J\Lambda((v_2 b v_3)_r)}(\sigma_t^\varphi(w_1^* w_2 a w_3)).\end{aligned}$$

Letting $r \longrightarrow \infty$, we get

$$\begin{aligned}\varphi^{(it)}_{w_1^* w_2 a w_3}(v_1^* v_2 b v_3) &= \omega_{J\Lambda(v_1), J\Lambda(v_2 b v_3)}(\sigma_t^\varphi(w_1^* w_2 a w_3)) \\ &= \varphi^{(-it)}_{v_1^* v_2 b v_3}(w_1^* w_2 a w_3). \quad \square\end{aligned}$$

To obtain the equality in Prop 2.7 for more general α, we use the idea of analytic continuation. The following proposition is crucial for this purpose.

Proposition 2.8. Let $\alpha, \beta \in \mathbb{C}$, and set $E = \{\gamma \in \mathbb{C} | \operatorname{Re}\alpha \leq \operatorname{Re}\gamma \leq \operatorname{Re}\beta\}$. Suppose that a function

$$f : E \to \mathcal{M}_*$$

satisfies the following four conditions:

(1) f is holomorphic in the interior of E.

(2) f is $\sigma(\mathcal{M}_*, \mathfrak{a}_0^2)$-continuous on E.

(3) f is uniformly continuous on the lines $\operatorname{Re} z = \operatorname{Re}\alpha$, $\operatorname{Re} z = \operatorname{Re}\beta$ with respect to the norm $\|\cdot\|_1$.

(4) f is norm bounded.

Then, f is norm continuous on D, that is,f belongs to $\mathcal{A}(E; \mathcal{M}_*)$.

Proof)
By applying a suitable affine transformation, we may assume that $E = D_{1/2} = \{\gamma \in \mathbb{C} | -1/2 \leq \operatorname{Re}\gamma \leq 1/2\}$
For $r > 0$, we set

$$\begin{aligned} f_r(z) &= \sqrt{\tfrac{r}{\pi}} \textstyle\int_{-\infty}^{\infty} e^{-rt^2} f(z+it)dt \in \mathcal{M}_*, \quad z \in D_{1/2}, \\ g_r(z) &= \sqrt{\tfrac{r}{\pi}} \textstyle\int_{-\infty}^{\infty} e^{-r(t+iz)^2} f(it)dt \in \mathcal{M}_*, \quad z \in \mathbb{C}, \end{aligned}$$

where each integrals are understood in the sense of Bochner. We will show that each f_r belongs to $\mathcal{A}(D_{1/2}; \mathcal{M}_*)$ and that f is the uniform limit of $\{f_r\}$, yielding that f also belongs to $\mathcal{A}(D_{1/2}; \mathcal{M}_*)$. The proof divides into several steps.
Step 1. We claim that f_r satisfies the conditions (1), (2), (3) and (4) in Proposition 2.8. Let γ be an arbitrary closed rectifiable curve in the interior of $D_{1/2}$. Then we compute

$$\begin{aligned} \int_\gamma \langle f_r(z), a\rangle_{\mathcal{M}_*,\mathcal{M}} dz &= \int_\gamma (\sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} \langle f(z+it), a\rangle_{\mathcal{M}_*,\mathcal{M}} dt) dz \\ &= \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} (\int_\gamma \langle f(z+it), a\rangle dz)_{\mathcal{M}_*,\mathcal{M}} dt \in \mathcal{M}_*, \ z \in D_{1/2} \\ &= 0 \end{aligned}$$

for all $a \in \Pi$ by Fubini's theorem and Cauchy's integral theorem. Hence f_r is holomorphic in the interior of $D_{1/2}$ by Morera's theorem.
To prove the $\sigma(\mathcal{M}_*, \Pi)$-continuity of f_r, we set, again for $a \in \Pi$,

$$k(z) = \langle f(z), a\rangle_{\mathcal{M}*,\mathcal{M}}, \ z \in D_{1/2}.$$

It follows that

$$|k(z)| \leq \|f(z)\|_1 \|a\|_\infty \leq M\|a\|_\infty, \ z \in D_{1/2},$$

where $M = \|f\|_{\mathcal{A}(D_{1/2},\mathcal{M}_*)} = \sup_{z \in D_{1/2}} \|f(z)\|_1$. The bounded convergence theorem applied for the finite measure $e^{-rt^2}dt$ tells us that

$$\int_{-\infty}^{\infty} e^{-rt^2} k(w+it)dt \to \int_{-\infty}^{\infty} e^{-rt^2} k(z+it)dt$$

as $w \in D_{1/2}$ approaches z. The uniform continuity of the adjoint of modular automorphism group σ_t^φ implies (3). We can easily prove (4).

Step 2. We claim that f_r admits an analytic continuation to the whole plane $\mathbb{C}$. For each $s \in \mathbb{R}$, we have

$$\begin{aligned} f_r(is) &= \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} f(is+it)dt \\ &= \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-r(t-s)^2} f(it)dt \\ &= g_r(is). \end{aligned}$$

Hence f_r and g_r coincide on the imaginary axis. By the same argument as in Step 1, we know that g_r is an entire function, so that g_r is the desired analytic continuation of f_r.

Step 3. We claim that $\{f_r\}$ converges to f uniformly on $D_{1/2}$ with respect to the norm $\|\cdot\|_1$ as $r \to \infty$.

Since f is uniformly continuous on the line $\operatorname{Re} z = 1/2$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|s_1 - s_2| \leq \delta \Longrightarrow \|f(1/2+is_1) - f(1/2+is_2)\|_1 < \varepsilon/3.$$

We take r so large that

$$\sqrt{\frac{r}{\pi}} \int_{\delta}^{\infty} e^{-rt^2} dt < \frac{\varepsilon}{8M}.$$

Then we estimate

$$\begin{aligned} &\|f(1/2+is) - f_r(1/2+is)\|_1 \\ = \;& \|\sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} f(1/2+is)dt - \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} f(1/2+is+it)dt\|_1 \\ \leq \;& \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rt^2} \|f(1/2+is) - f(1/2+is+it)\|_1 dt \\ = \;& \sqrt{\frac{r}{\pi}} (\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}) e^{-rt^2} \|f(1/2+is) - f(1/2+is+it)\|_1 dt \\ \leq \;& \sqrt{\frac{r}{\pi}} (\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}) e^{rt^2} \cdot 2M dt + \sqrt{\frac{r}{\pi}} \int_{-\delta}^{\delta} e^{-rt^2} \|f(1/2+is) - f(1/2+is+it)\|_1 dt \\ \leq \;& \varepsilon/3 \int_{-\delta}^{\delta} e^{-rt^2} dt \\ \leq \;& \frac{5}{6}\varepsilon < \varepsilon \end{aligned}$$

Hence f is the uniform limit of $\{f_r\}$ and we conclude that f belongs to $\mathcal{A}(D_{1/2}, \mathcal{M}_*)$ by the completeness of $\mathcal{A}(D_{1/2}, \mathcal{M}_*)$. $\square$

Proposition 2.9. Let α, β be complex numbers such that $\operatorname{Re} \alpha < \operatorname{Re} \beta$. If $a \in L_{(\alpha)} \cap L_{(\beta)}$, then we have $a \in L_{(\gamma)}$ for all $\gamma \in \mathbb{C}$, $\operatorname{Re} \alpha \leq \operatorname{Re} \gamma \leq \operatorname{Re} \beta$.

To prove Proposition 2.9, we use a classical result in harmonic analysis:

Lemma 2.10. (See [**BL**]) There exist two integrable continuous functions

$$K_j : D^\circ \times \mathbb{R} \to \mathbb{R}_{\geq 0}, \; j = 0, 1$$

such that for any function $f \in \mathcal{A}(D)$ we have the presentation

$$f(z) = \int_{-\infty}^{\infty} f(it) K_0(z,t)dt + \int_{-\infty}^{\infty} f(1+it) K_1(z,t)dt$$

for all z in the interior of E.

Proof of Proposition 2.9)

Fix $x, y \in \mathfrak{a}_0$ and consider the function

$$\varphi_{x,y} : E \to \mathbb{C},$$

$$\varphi_{x,y}(\gamma) = (aJ\Delta^{\overline{\gamma}}\Lambda(x)|J\Delta^{-\gamma}\Lambda(y)),\ \gamma \in E,$$

where $E = \{\gamma \in \mathbb{C}|\ \mathrm{Re}\,\alpha \leq \mathrm{Re}\,\gamma \leq \mathrm{Re}\,\beta\}$.

Then, by Lemma 2.10, and applying some suitable change of variables, there exist two integrable continuous functions K_0, K_1 such that

$$\varphi_{x,y}(\gamma) = \int_{-\infty}^{\infty} \varphi_{x,y}(\alpha + it)K_0(\gamma, t)dt + \int_{-\infty}^{\infty} \varphi_{x,y}(\beta + it)K_1(\gamma, t)dt$$

for all γ in the interior of E. Now, we define an $\mathcal{M}_*$-valued function Φ on E by using Bochner integrals.

$$\Phi(\gamma) = \begin{cases} \int_{-\infty}^{\infty} K_0(\gamma,t)\varphi_a^{(\alpha+it)}dt + \int_{-\infty}^{\infty} K_1(\gamma,t)\varphi_a^{(\beta+it)}dt. & \text{if } \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma < \mathrm{Re}\,\beta \\ \varphi_a^{(\gamma)} & \text{if } \gamma \in \partial E \end{cases}$$

These integrals exist and indeed define elements of $\mathcal{M}_*$ by Proposition 2.6 (2) together with the fact that the adjoint action of modular automorphisms on $\mathcal{M}_*$ is pointwise-norm continuous. We will show that $\Phi \in \mathcal{A}(E; \mathcal{M}_*)$.

For $x, y \in \mathfrak{a}_0$, we have

$$\begin{aligned}
&\langle\Phi(\gamma), x^*y\rangle_{\mathcal{M}_*,\mathcal{M}} \\
&= \int_{-\infty}^{\infty} K_0(t,\gamma)\langle\varphi_a^{(\alpha+it)}, x^*y\rangle_{\mathcal{M}_*,\mathcal{M}}dt + \int_{-\infty}^{\infty} K_1(t,\gamma)\langle\varphi_a^{(\beta+it)}, x^*y\rangle_{\mathcal{M}_*,\mathcal{M}}dt \\
&= \int_{-\infty}^{\infty} K_0(t,\gamma)(aJ\Delta^{\overline{\alpha}-it}\Lambda(x)|J\Delta^{-\alpha-it}\Lambda(y))dt + \int_{-\infty}^{\infty} K_1(t,\gamma)(aJ\Delta^{\overline{\beta}-it}\Lambda(x)|J\Delta^{-\beta-it}\Lambda(y))dt \\
&= \int_{-\infty}^{\infty} \varphi_{x,y}(\alpha+it)K_0(t,\gamma)dt + \int_{-\infty}^{\infty} \varphi_{x,y}(\beta+it)K_1(t,\gamma)dt \\
&= \varphi_{x,y}(\gamma).
\end{aligned}$$

for all γ in the interior of E. This formula means that the function Φ is analytic in the interior and $\sigma(\mathcal{M}_*, \mathfrak{a}_0^2)$-continuous on E. Considering the maps

$$t \in \mathbb{R} \mapsto \varphi^(\alpha + it) \in \mathcal{M}_*,$$

and

$$t \in \mathbb{R} \mapsto \varphi^(\beta + it) \in \mathcal{M}_*,$$

they are both periodic function of period 2π. Hence Φ satisfies Conditions (1), (2), (3) and (4) in Proposition 2.8, so that $\Phi \in \mathcal{A}(E; \mathcal{M}_*)$. Consequently, for all $\gamma \in E$, a belongs to $L_{(\gamma)}$ and $\varphi^(\gamma) = \Phi(\gamma)$. □

Proposition 2.11. Let α be a complex number such that $\mathrm{Re}\,\alpha > 0$. Then, for any $a \in L_{(\alpha)}$ and for any $w_1, w_2, w_3 \in \mathfrak{a}_0$, we have

$$w_1^* w_2 a w_3 \in L_{(0)} \cap L_{(\alpha)}.$$

Proof)

This follows easily from Propositions 2.6(1) and 2.7. □

Proposition 2.12. Let α be any complex number and let $a \in L_{(\alpha)}$, $b \in L_{(-\alpha)}$, $w_j, v_j \in \mathfrak{a}_0$, $j = 1,2,3$. Then we have

$$\varphi^{(\alpha)}_{w_1^* w_2 a w_3}(v_1^* v_2 b v_3) = \varphi^{(-\alpha)}_{v_1^* v_2 b v_3}(w_1^* w_2 a w_3).$$

Proof)

This has been already shown in the case of $\operatorname{Re}\alpha = 0$, and by symmetry, it suffices to show when $\operatorname{Re}\alpha > 0$

We set

$$E_\alpha = \{\beta \in \mathbb{C} | 0 \le \operatorname{Re}\beta \le \operatorname{Re}\alpha\},$$

and consider the function

$$\mu : E_\alpha \to \mathbb{C}$$

defined by

$$\mu(\beta) = \varphi^{(\beta)}_{w_1^* w_2 a w_3}(v_1^* v_2 b v_3) - \varphi^{(-\beta)}_{v_1^* v_2 b v_3}(w_1^* w_2 a w_3), \ \beta \in E_\alpha.$$

By Proposition 2.9, μ is well-defined, and by Proposition 2.8, μ belongs to $\mathcal{A}(E_\alpha)$. Proposition 2.7 tells us that

$$\mu(it) = 0$$

for all $t \in \mathbb{R}$, then we have μ is identically zero by the Phragmén-Lindelöf theorem. Hence we get the desired formula. □

The next lemma is used to complete the proof of Theorem 2.4.

Lemma 2.13. ([Te], Lemma 9) For any $\delta > 0$, there exists a net $\{e_j\} \in \mathfrak{a}_0$ such that

(a) $\|\sigma^\varphi_\alpha(e_j)\|_\infty \le e^{\delta|\operatorname{Im}\alpha|^2}$, $\alpha \in \mathbb{C}$, for all j.

(b) $e_j \to 1$ strongly.

Proof of Theorem 3.4)

Let $x \in L_{(\alpha)}$, $y \in L_{(-\alpha)}$. We take $\{e_j\}$ as in the previous lemma ($\delta = 1$, say). Set

$$\begin{aligned} x_j &= \sigma^\varphi_{i\alpha+i/2}(e_j)^2 x \sigma^\varphi_{i\alpha-i/2}(e_j) \quad , \\ y_j &= \sigma^\varphi_{i\alpha+i/2}(e_j)^2 y \sigma^\varphi_{i\alpha-i/2}(e_j) \quad . \end{aligned}$$

Then by Proposition 3.12, we have

$$\varphi^{(\alpha)}_{x_j}(y_j) = \varphi^{(-\alpha)}_{y_j}(x_j). \tag{2.1}$$

By Proposition 2.6(a), we have

$$\varphi^{(\alpha)}_{x_j} = e_j^2 \varphi^{(\alpha)}_x e_j,$$

$$\varphi^{(-\alpha)}_{y_j} = e_j^2 \varphi^{(-\alpha)}_y e_j.$$

Since the bounded net $\{e_j\}$ converges to 1 strongly, we have

$$\begin{aligned} &\|\varphi^{(\alpha)}_{x_j} - \varphi^{(\alpha)}_x\|_1 \to 0, \\ &\|\varphi^{(-\alpha)}_{y_j} - \varphi^{(-\alpha)}_y\|_1 \to 0, \end{aligned}$$

Hence, letting j to infinity in the formula (2.1), we have

$$\varphi^{(\alpha)}_x(y) = \varphi^{(-\alpha)}_y(x).$$

because both $\|x_j\|_\infty$, $\|y_j\|_\infty$ are bounded. This concludes the proof of Theorem 2.4. □

Corollary 2.14.

$$j^*_{(-\alpha)}(\mathcal{M}) \cap i^*_{(-\alpha)}(\mathcal{M}_*) = L_{(\alpha)}, \quad \alpha \in \mathbb{C}.$$

Proof)

By Theorem 2.4, $L_{(\alpha)}$ can be regarded as a subspace of $L^*_{(-\alpha)}$. Suppose that $j^*_{(-\alpha)}(x) = i^*_{(-\alpha)}(\psi)$. Then for any $y, z \in \mathfrak{a}_0$, we have

$$\begin{aligned}\psi(y^*z) &= \varphi^{(-\alpha)}_{y^*z}(x) \\ &= (xJ\Delta^{\overline{\alpha}}\Lambda(y)|J\Delta^{-\alpha}\Lambda(z)).\end{aligned}$$

Hence $x \in L_{(\alpha)}$ and $\varphi^{(\alpha)}_x = \psi$. □

3. Equivalence of Non-commutative L^p-spaces

In this section, we will prove that $L^p_{(\alpha)}(\varphi)$ is isometrically isomorphic to $L^p_{(\beta)}(\varphi)$, $\alpha, \beta \in \mathbb{C}$, $1 < p < \infty$. In particular, $L^p_{(\alpha)}$ is isometrically isomorphic to Terp's L^p-space $L^p_{(0)}(\varphi)$.

To this end, we use the weaker interpolation $\mathcal{F}'(\mathcal{M}, \mathcal{M}_*)_{(\alpha)}$ with $\Pi = \mathfrak{a}_0^2$. The following proposition is an improvement of Kosaki's.

Proposition 3.1. Assume that the unit ball of A_0 is $\sigma(\Sigma(A), \Pi)$-complete in $\Sigma(A)$. Let Y be a reflexive Banach space satisfying

$$A_0 \cap A_1 \subset Y \subset \Sigma(A)$$

as a linear space. Then $C_\theta(A) = Y = C'_\theta(A)$ provided that the following two conditions are fulfilled:

(1) for each $y \in Y$ there exists an $f \in \mathcal{F}(A)$ such that

$$f(\theta) = y, \ \|f\|_{\mathcal{F}'(A)} = \|y\|_Y,$$

(2) each $g \in \mathcal{F}_0(A)$ satisfies

$$\|g(\theta)\|_Y \leq \|g\|_{\mathcal{F}(A)},$$

where $\mathcal{F}_0(A) = \{f \in \mathcal{F}(A) | f(z) = \exp(\varepsilon z^2)\sum_{j=1}^N a_j \exp(\lambda_j z),\ z \in D,\ \varepsilon > 0,\ a_j \in A_0 \cap A_1,\ \lambda_j \in \mathbb{R}\}$.

Proof)

The proof proceeds similarly as that of [**Ko**,Theorem 1.8]. Refer also to the proof of 2.8. □

As an immediate consequence of Proposition 1.1, we have the following:

Corollary 3.2. Let A, B be two compatible pairs. Assume that the unit ball of B_0 is $\sigma(\Sigma(B), \Pi)$-complete in $\Sigma(B)$, where Π is a $\sigma(\Sigma(B)^*, \Sigma(B))$-subspace of $\Sigma(B)^*$. Moreover, suppose that A_θ is reflexive for all θ, $0 < \theta < 1$. If there is a isometric map of $\mathcal{F}'(A)$ into $\mathcal{F}'(B)$, then $A_\theta = B_\theta$ (equal norms), $0 < \theta < 1$.

Theorem 3.3. Let α be a real number. Then $L^p_{(\alpha)}(\varphi)$ is isomerically isomorphic to $L^p_{(0)}(\varphi)$ for $1 < p < \infty$.

Proof)

By Corollary 3.2, it suffices to construct a isometric map $\Phi : \mathcal{F}'_{(0)} \to \mathcal{F}'_{(\alpha)}$. Let $f \in \mathcal{F}'_{(0)}$. Then we define

$$(\Phi f)(z) = \begin{cases} \int_{-\infty}^{\infty} j^*_{(-\alpha)}(\sigma^{\varphi}_{t\alpha} f(it)) K_0(z,t)dt + \int_{-\infty}^{\infty} i^*_{(-\alpha)}(f(1+it)\circ\sigma^{\varphi}_{-t\alpha})K_1(z,t)dt & 0 < \operatorname{Re} z < 1 \\ j^*_{(-\alpha)}(\sigma^{\varphi}_{t\alpha} f(it)) & z = it, t \in \mathbb{R} \\ i^*_{(-\alpha)}(f(1+it)\circ\sigma^{\varphi}_{-t\alpha}) & z = 1+it, t \in \mathbb{R}. \end{cases}$$

We will show that $\Phi f \in \mathcal{F}'_{(\alpha)}$. For $y, w \in \mathfrak{a}_0$, we set

$$\eta_{y^*w}(z) = \langle f(z), (\sigma^{\varphi}_{i\alpha(1-\bar{z})}(y))^* \sigma^{\varphi}_{-i\alpha(1-z)}(w)) \rangle_{\Sigma_{(0)}, L_{(0)}}, \ z \in D.$$

It is easy to see that η_{y^*w} belongs to $\mathcal{A}(D)$. Then we compute

$$\begin{aligned} \eta_{y^*w}(it) &= \langle f(it), (\sigma^{\varphi}_{i\alpha(1+it)}(y))^* \sigma^{\varphi}_{-i\alpha(1-it)}(w)) \rangle_{\Sigma_{(0)}, L_{(0)}} \\ &= (f(it) J \Delta^{-\alpha(1+it)} \Lambda(y) | J \Delta^{\alpha(1-it)} \Lambda(w)) \\ &= (\sigma^{\varphi}_t(f(it)) J \Delta^{-\alpha} \Lambda(y) | J \Delta^{\alpha} \Lambda(w)) \\ &= \langle j^*_{(\alpha)}(f(it)), y^*w \rangle_{\Sigma_{(\alpha)}, L_{(\alpha)}} \\ &= \langle (\Phi f)(it), y^*w \rangle_{\Sigma_{(\alpha)}, L_{(\alpha)}}, \end{aligned}$$

and

$$\begin{aligned} \eta_{y^*w}(1+it) &= \langle f(1+it), (\sigma^{\varphi}_{i\alpha(it)}(y))^* \sigma^{\varphi}_{-i\alpha(-it)}(w)) \rangle_{\Sigma_{(0)}, L_{(0)}} \\ &= \langle f(1+it) \circ \sigma^{\varphi}_{-\alpha t}, y^*w \rangle_{\mathcal{M}_*, \mathcal{M}} \\ &= \langle (\Phi f)(1+it), y^*w \rangle_{\Sigma_{(\alpha)}, L_{(\alpha)}}. \end{aligned}$$

By Lemma 2.10, we have $\Phi f \in \mathcal{F}_{(\alpha)}$ with

$$\langle (\Phi f)(z), y^*w \rangle_{\Sigma_{(\alpha)}, L_{(\alpha)}} = \eta_{y^*w}(z), \ z \in D.$$

We also easily check that $\|f\|_{\mathcal{F}_{(0)}} = \|\Phi f\|_{\mathcal{F}'_{(\alpha)}}$, since the difference factor on the boundary is just modular automorphisms. Hence we completed the proof. □

Theorem 3.4. Let α, β be complex numbers such that $\operatorname{Re}\alpha = \operatorname{Re}\beta$. Then $L^p_{(\alpha)}$ is isometrically isomorphic to $L^p_{(\beta)}$, $1 < p < \infty$.

Proof)

We may assume that α is real. Then by the previous theorem, $L^p_{(\alpha)}$ is reflexive. Again this time, we will construct an isometric map from $\mathcal{F}'_{(\alpha)}$ into $\mathcal{F}'_{(\beta)}$.

Let $f \in \mathcal{F}'_{(\alpha)}$. Then we define

$$(\Phi f)(z) = \begin{cases} \int_{-\infty}^{\infty} j^*_{(-\beta)}(\sigma^{\varphi}_s(f(it))) K_0(z,t)dt + \int_{-\infty}^{\infty} i^*_{(-\beta)}(f(1+it))K_1(z,t)dt & 0 < \operatorname{Re} z < 1 \\ j^*_{(-\beta)}(\sigma^{\varphi}_s(f(it))) & z = it, t \in \mathbb{R} \\ i^*_{(-\beta)}(f(1+it)) & z = 1+it, t \in \mathbb{R}, \end{cases}$$

where $s = -i(\beta - \alpha)$. We will show that $\Phi f \in \mathcal{F}'_{(\beta)}$. For $y, w \in \mathfrak{a}_0$, we compute

$$\begin{aligned} \langle (\Phi f)(it), y^*w \rangle_{\Sigma_{(\beta)}, L_{(\beta)}} &= \varphi^{(\beta)}_{y^*w}(\sigma^{\varphi}_s(f(it))) \\ &= \varphi^{(\alpha)}_{y^*w}(f(it)) \text{ by Proposition 2.6(b)} \\ &= \langle f(it), y^*w \rangle_{\Sigma_{(\alpha)}, L_{(\alpha)}} \end{aligned}$$

and

$$\langle (\Phi f)(1+it), y^*w\rangle_{\Sigma_{(\beta)},L_{(\beta)}} = \langle f(1+it), y^*w\rangle_{\mathcal{M}_*,\mathcal{M}}.$$

By Lemma 2.10, we have $\Phi f \in \mathcal{F}'_\beta$ with

$$\langle (\Phi f)(z), y^*w\rangle_{\Sigma_{(\beta)},L_{(\beta)}} = \eta_{y^*w}(z), \ z \in D.$$

Moreover, this map is isometric because the difference factor on the boundary (difference arises only on the left boundary line) is just modular automorphisms. Hence we completed the proof. □

Combining Theorems 3.3 and 3.4, we obtain the following result:

Corollary 3.5. Let α, β be any two complex numbers. Then $L^p_{(\alpha)}$ is isometrically isomorphic to $L^p_{(\beta)}$, $1 < p < \infty$.

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